Final Exam — Group Theory (WIGT-07)

Monday January 22, 2017, 9:00h–12.00h

University of Groningen

Instructions

1. Write your name and student number on every page you hand in.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. Your grade for this exam is $(P+10)/10$, where P is the number of points for this exam.

Problem 1 (15 points)

a) Give the definition of a normal subgroup of a group.

Solution: A subgroup H of a group G is normal if $aH = Ha$ for all $a \in G$. (5 points)

b) Write down Lagrange's theorem.

Solution: Let H be a subgroup of a finite group G. Then $\#H$ divides $\#G$. (5 points)

c) Give the definition of a finitely generated abelian group.

Solution: A group $(G, \cdot, e)$ is abelian if $a \cdot b = b \cdot a$ for all $a, b \in G$, It is finitely generated if there are $x_1, \ldots, x_n \in G$ such that every $x \in G$ can be written as $x = x_{i_1}^{\pm 1} \cdots x_{i_m}^{\pm 1}$ with $i_1, \ldots, i_m \in \{1, \ldots, n\}$ (5 points)

Problem 2 (15 points)

Let $\tau = (5\,6\,7\,9\,8)(3\,4\,5\,6)(2\,3\,4\,5)(1\,2\,7) \in S_9$.

a) Compute the order of τ.

Solution: We compute the decomposition of τ into disjoint cycles and find $\tau = (1\,4\,7)(2\,9\,8\,5)(3\,6)$ (3 points). Hence $\mathrm{ord}(\tau) = \mathrm{lcm}(3 \cdot 4 \cdot 2) = 12$, because the order of a product of disjoint cycles is the least common multiple of the lengths of the cycles. (2 points)

b) Compute the sign of τ.

Solution: By multiplicativity of the sign or by using the formula for the sign of a product of cycles (1 point), we find it is $(-1)^{5-1+4-1+4-1+3-1} = 1$ (using the definition of τ) or $(-1)^{3-1+4-1+2-1} = 1$ (using the decomposition into disjoint cycles). (2 points)

c) Find the number of elements of the conjugacy class of τ.

Solution: The conjugacy class of a permutation is determined completely by the decomposition into disjoint cycles, so the desired number of elements is the number of $\sigma = (i_1\,i_2)(j_1\,j_2\,j_3)(k_1\,k_2\,k_3\,k_4) \in S_9$, where the three cycles are disjoint. (2 points) The number of k-cycles in S_n is $\frac{n!}{k(n-k)!}$ (1 point). Hence there are $\frac{9!}{2(9-2)!} = 36$ choices for $(i_1\,i_2)$.

After these are fixed, there are $\frac{7!}{3(7-3)!} = 70$ choices for $(j_1\, j_2\, j_3)$, and once this is fixed as well, 6 choices for $(k_1\, k_2\, k_3\, k_4)$. (3 points) Therefore the conjugacy class of τ contains $36 \cdot 70 \cdot 6 = 15120$ elements (1 point).

Problem 3 (20 points)

Let G be a group and let $X \subseteq G$ be non-empty. We define

$$N_G(X) := \left\{a \in G : aXa^{-1} = X\right\} .$$

a) Show that $N_G(X)$ is a subgroup of G.

Solution: It suffices to check conditions (H1 – H3) from the lectures. (2 points) (H1): $eXe^{-1} = X$, so $e \in N_G(X)$ (1 point). (H2): $a, b \in N_G(X) \Rightarrow (ab)X(ab)^{-1} = a(bXb^{-1})a = aXa^{-1} = X$, so $ab \in N_G(X)$ (2 points) (H3): $a \in N_G(X) \Rightarrow aXa^{-1} = X \Rightarrow a^{-1}Xa = X \Rightarrow a^{-1} \in N_G(X)$ (2 points)

b) Show that

$$\#\{aXa^{-1} : a \in G\} = [G : N_G(X)].$$

Solution: We need to show that for $a, b \in G$, we have $aXa^{-1} = bXb^{-1}$ iff $aX = bX$. Now $aXa^{-1} = bXb^{-1} \Leftrightarrow X = b^{-1}aXa^{-1}b = b^{-1}aX(b^{-1}a)^{-1} \Leftrightarrow b^{-1}a \in N_G(X) \Leftrightarrow aN_G(X) = bN_G(X)$ (6 points)

c) Find a group G and a non-empty subset X of G such that $N_G(X)$ is not a normal subgroup of G.

Solution: Let $G = S_3$ and let $X = \{\tau\}$, where $\tau = (12) \in G$ (2 points). Obviously $\mathrm{id}, \tau \in N_G(X)$ (1 point) One computes $\sigma\tau\sigma^{-1} \notin X$ for all $\sigma \in G \setminus \{\mathrm{id}, \tau\}$ (2 points), so $N_G(X)$ is the subgroup $H = \langle\tau\rangle$ (1 point). By the same computation, H is not normal. (1 points)

Of course, there are other examples; these are graded similarly (there are 7 points for this subproblem)

Problem 4 (10 points)

Let G be a group of order 48. Show that G is not simple.

Solution: We have $48 = 2^4 \cdot 3$. (1 point) For a prime $p \mid 48$, let N_p be the number of Sylow-2 groups in G. If we find $N_p = 1$ for some p, then we know that the unique Sylow p-group in G is normal and, since it has order p, it is not G or $\{e\}$, so G is not simple. (2 points)

We have $N_2 \equiv 1 \pmod 2$ and $N_2 \mid 3$, so $N_2 \in \{1, 3\}$. So either $N_2 = 1$, in which case we're done, or $N_2 = 3$ (3 points). In the latter case, let X denote the set of Sylow-2 groups. We get a homomorphism $\varphi : G \to S_X$ given by sending $a \in G$ to conjugation γ_a by a (since $\gamma_{ab} = \gamma_a \circ \gamma_b$ for all $a, b \in G$) (2 points). The kernel of φ is a normal subgroup of G, so if G is simple, it has to either consist of $\{e\}$ (impossible, since $\#G = 48 > 6 = \#S_X$) or all of G. This means that all Sylow 2-groups are fixed by conjugation, which is not the case by Sylow's Theorem (2 points). Hence G is simple.

Problem 5 (10 points)

Let n be a positive integer and consider the subgroup $\mu_n := \{z \in \mathbb{C}^\times : z^n = 1\}$ of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (where the group law is multiplication). Show that

$$\mathbb{C}^\times/\mu_n \cong \mathbb{C}^\times.$$

Solution: Let $f_n : \mathbb{C}^\times \to \mathbb{C}^\times$ be defined by $f_n(z) := z^n$. (2 points) Then $f_n(zw) = (zw)^n = z^n w^n = f_n(z)f_n(w)$, so f_n is a homomorphism. (2 points) It is surjective, since every complex number has an nth root by the fundamental theorem of algebra. (2 points) The kernel of f_n is precisely μ_n (1 point), so the first isomorphism theorem implies (3 points)

$$\mathbb{C}^\times/\mu_n = \mathbb{C}^\times/\ker(f_n) \cong f_n(\mathbb{C}^\times) = \mathbb{C}^\times.$$

Problem 6 (20 points)

Let $H \subset \mathbb{Z}^3$ be generated by $(2, 0, 2), (6, 6, 6)$ and $(8, 36, 38)$.

a) Find a basis of H.

Solution: Let A denote the matrix with columns equal to the given generators of H. Then $\det(A) = 360 \neq 0$, so the given generators form a basis. (Alternatively, run the algorithm in b) and see that the rank of H is 3, which implies the same result) (6 points)

b) Find the rank and the elementary divisors of $\mathbb{Z}^3/H$.

Solution: We apply the algorithm from the lecture (2 points) to transform A into the diagonal matrix with entries $30, 6, 2$. (8 points) Hence we find $\mathbb{Z}^3/H \cong \mathbb{Z}/30\mathbb{Z}/ \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2/\mathbb{Z}$ (2 points), so that the rank is 0 (1 point) (we can already conclude that from part a) and that the elementary divisors are $30, 6, 2$. (1 point)

End of test (90 points)